Universal Sets of n Points for 1-bend Drawings of Planar Graphs with n Vertices *

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Abstract. This paper shows that any planar graph with n vertices can be point-set embedded with at most one bend per edge on a universal set of n points in the plane. An implication of this result is that any number of planar graphs admit a simultaneous embedding without mapping with at most one bend per edge.

1 Introduction

Let S be a set of m distinct points in the plane and let G be a planar graph with n vertices ($n \leqslant m$). A *point-set embedding* of G on S is a planar drawing of G such that each vertex is drawn as a point of S and the edges are drawn as poly-lines. The problem of computing point-set embeddings of planar graphs has a long tradition both in the graph drawing and in the computational geometry literature (see, e.g., [5, 6, 8]). Considerable attention has been devoted to the study of universal sets of points. A set S of m points is said to be *h-bend universal* for the family of planar graphs with n vertices ($n \leqslant m$) if any graph in the family admits a point-set embedding onto S that has at most h bends along each edge.

Gritzman, Mohar, Pach and Pollack [5] proved that every set of n distinct points in the plane is 0-bend universal for the all outerplanar graphs with n vertices. De Fraysseix, Pach, and Pollack [3] and independently Schnyder [9] proved that a grid with $O(n^2)$ points is 0-bend universal for all planar graphs with n vertices. De Fraysseix et al. [3] also showed that a 0-bend universal set of points for all planar graphs having n vertices cannot have $n + o(\sqrt{n})$ points. This last lower bound was improved by Chrobak and Karloff [2] and later by Kurowski [7] who showed that linearly many extra points are necessary for a 0-bend universal set of points for all planar graphs having n vertices. On the other hand, if two bends along each edge are allowed, a tight bound on the size of the

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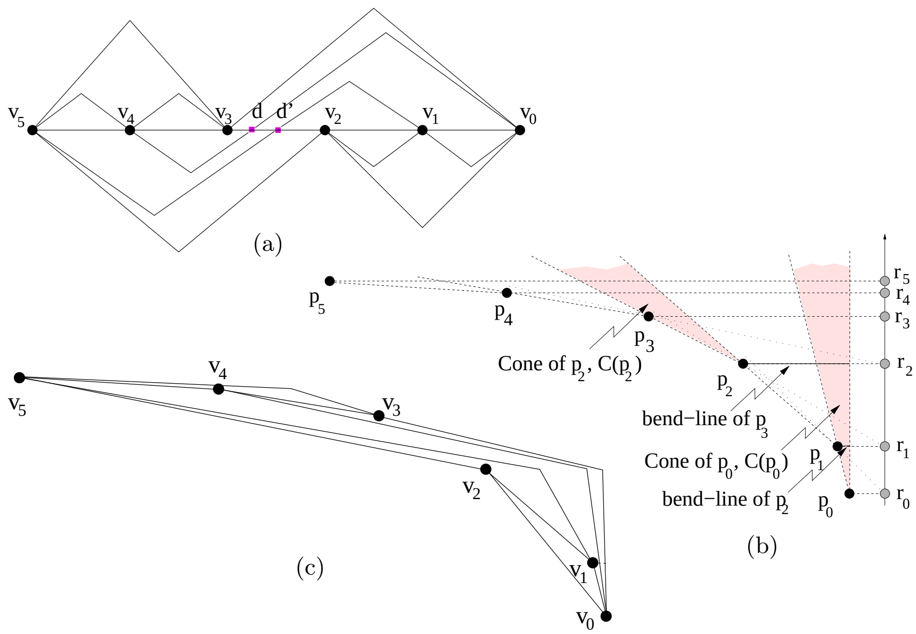

Fig. 1. (a) A proper monotone topological book embedding. The spine crossings d and d' are proper. (b) A necklace of six points. The cone of p_0, the cone of p_2, the bend-line of p_2, and the bend-line of p_3 are highlighted. (c) A point-set embedding computed by Algorithm `1-bend Universal Drawer`.

point-set is known: Kaufmann and Wiese [6] proved that every set of n distinct points in the plane is 2-bend universal for all planar graphs with n vertices.

In this paper we study the minimum size of a universal set of points for all planar graphs with n vertices under the assumption that at most one bend per edge is allowed in the point-set embedding. We prove the following theorem.

Theorem 1. *Let $\mathcal{F}_n$ be the family of all planar graphs with n vertices. There exists a set of n distinct points in the plane that is* 1*-bend universal for $\mathcal{F}_n$.*

The proof is constructive; an example is shown in Figure 1. We define a set S of n points and show how to compute an embedding of any planar graph with n vertices on S such that the resulting drawing has at most one bend per edge. The drawing procedure starts by computing a special type of book embedding defined in Section 2, and then uses this book embedding to construct the point-set embedding with the algorithm described in Section 3.

Our universal set of n points can be defined either (i) with algebraic coordinates such that they are the vertices of a convex chain with unit-length edges or (ii) on a regular grid of size $n2^n$ by n. In the former case all planar graphs of $\mathcal{F}_n$ can be drawn on that point set with all bend-points and vertices in a square of size n by n at distance at least $\frac{1}{2d}$ apart, where d is the maximum degree of the graph. In the latter case, the graphs can be drawn with all bend-points on the grid points of the $n2^n$ by n grid.

We conclude this introduction by noting a result that is immediately implied by Theorem 1. Two planar graphs G_1 and G_2 with the same set of vertices are said to admit a *simultaneous embedding without mapping* if there exists a set of points in the plane that supports a point-set embedding of both G_1 and G_2 [1]. It is not known whether any two planar graphs admit a simultaneous embedding without mapping such that all edges are straight-line segments. A consequence of [5] is that a planar graph has a straight-line simultaneous embedding without mapping with any number of outerplanar graphs. A consequence of [6] is that any two planar graphs have a simultaneous embedding without mapping such that each edge is drawn with at most two bends. Theorem 1 implies the following.

Corollary 1. *Any number of planar graphs sharing the same vertex set admit a simultaneous embedding without mapping with at most one bend per edge.*

2 Monotone Topological Book Embeddings

Consider the Cartesian coordinate system (O, x, y) and let p, q be two points in the plane. We say that p is *left of* q and we denote it as $p < q$ if the x-coordinate of p is less than the x-coordinate of q; we shall also use the notation $p \leqslant q$ to mean that either p is left of q or p coincides with q; we define similarly $p > q$ and $p \geqslant q$. A *spine* is a horizontal line. Let ℓ be a spine and let p, q be two points of ℓ. Let $p < q$ and let b be a point of the perpendicular bisector of $\overline{pq}$, at positive distance from ℓ. An *arc* connecting p to q, denoted as (p, q), is a polygonal chain consisting of two segments: segment $\overline{pb}$ and segment $\overline{bq}$. Point p is the *left endpoint* of (p, q), point q is the *right endpoint* of (p, q), and point b is the *bend-point* of (p, q). Arc (p, q) can be either in the half-plane above the spine or in the half-plane below the spine (such half-planes are assumed to be closed sets); in the first case we say that the arc is in the *top page* of ℓ, otherwise it is in the *bottom page* of ℓ. From now on, when we denote an arc as (p, q) we shall implicitly assume that p is its left endpoint.

Let $G = (V, E)$ be a planar graph. A *monotone topological book embedding* of G, denoted Γ, is a planar drawing such that all vertices of G are represented as points of a spine ℓ and each edge is either represented as an arc in the bottom page, or as an arc in the top page, or as a poly-line that crosses the spine and consists of two consecutive arcs. Let $e = (u, v)$ be an edge of a monotone topological book embedding that crosses the spine at a point d; assuming that u is left of v along the spine, e is such that: (i) $u < d < v$, (ii) arc (u, d) is in the bottom page, and (iii) arc (d, v) is in the top page. Point d is called the *spine crossing* of (u, v). Refer to Figure 1(a). Also, let u' be the rightmost vertex along the spine of Γ such that $u' < d$ and let v' be the leftmost vertex of the spine of Γ such that $d < v'$. We say that u' and v' are the two *bounding vertices* of d. We say that d is a *proper spine crossing* if its bounding vertices u' and v' are such that $u < u' < d < v' < v$. (The spine crossings d and d' of Figure 1(a) are both proper and both bounded by v_3 and v_2). A monotone topological book embedding is *proper* if all of its spine crossings are proper. Di Giacomo et al. [4] proved that, for every planar graph, a monotone topological book embedding exists and can be computed (in linear time in the size of the graph). Since an edge that crosses the spine with a non-proper spine crossing can be replaced by a single arc, we obtain the following lemma.

Lemma 1. *Every planar graph has a proper monotone topological book embedding which can be computed in linear time in the size of the graph.*

Let now Γ be a proper monotone topological book embedding of a planar graph G. If we insert a dummy vertex for each spine crossing of Γ, we obtain a new topological book embedding Γ' such that Γ' represents a planar subdivision G' of G obtained by splitting with a vertex some of the edges of G. We call the graph G' an *augmented form* of G and the drawing Γ' an *augmented topological book embedding* of G. A vertex of G' that is also a vertex of G is called a *real vertex* of Γ'; a vertex of G' that corresponds to a spine crossing of Γ is called a *division vertex* of Γ'. Note that every division vertex of Γ' has degree two and that every edge of Γ' is either an arc in the top page or an arc in the bottom page. The *bounding vertices of a division vertex* d of Γ' are the two real vertices that form the bounding vertices of the spine crossing corresponding to d in Γ. The following property is a consequence of the planarity of Γ'.

Property 1. Let $a = (u, v)$ and $a' = (u', v')$ be two distinct arcs of Γ' that are in the same page and such that $u < u'$. Then, (i) $u < v \leqslant u' < v'$ or (ii) $u < u' < v' \leqslant v$.

3 Proof of Theorem 1

We prove Theorem 1 by first defining a family of sets of n points in convex position (Subsection 3.1) and then by describing an algorithm that computes a point-set embedding of any planar graph with n vertices on the n-point element of the family (Subsection 3.2).

3.1 Necklaces, Cones, and Bend-lines

Let p_0 be any point on the x-axis strictly left of O and p_1 be any point strictly in the top-left quadrant of p_0. We construct p_{i+2}, for $0 \leqslant i \leqslant n-2$, from p_i and p_{i+1} as follows. Let r_i be the projection of p_i on the vertical y-axis. Point p_{i+2} can be chosen anywhere on or below the line through r_i and p_{i+1} and strictly above the horizontal line through p_{i+1}. Let S be any set of n points defined by the above procedure; we call S a *necklace* of n points. See Figure 1(b).

The *cone of* p_0, denoted as $C(p_0)$, is the wedge with apex p_0 and bounded by the vertical half-line above p_0 and by the ray emanating from p_0 and through p_1. The *cone of* p_i $(1 \leqslant i \leqslant n-2)$, denoted as $C(p_i)$, has p_i as its apex and is bounded by two rays emanating from p_i with directions $\overrightarrow{p_{i-1}p_i}$ and $\overrightarrow{p_i p_{i+1}}$. In what follows we assume that $C(p_i)$ is an open set $(0 \leqslant i \leqslant n-1)$.

The *bend-line* of p_i $(i > 1)$ is the relatively-open horizontal segment from p_{i-1} to the vertical line through p_0. The following properties follow from the definition of a necklace and can be proved with elementary geometric arguments. Let $S = \{p_0, p_1, \ldots p_{n-1}\}$ be a necklace of n points and let $CH(S)$ be its convex hull. Note that $p_0, \ldots, p_{n-1}$ are ordered from right to left, *i.e.*, $p_{n-1} < \ldots < p_0$.

Property 2. Let $p_h < p_t$ $(t > 1)$ be two points of S and let q be a point on the bend-line of p_t. Segments $\overline{p_h q}$ and $\overline{p_t p_{t-1}}$ intersect in their relative interior.

Property 3. Let $p_{h'} \leqslant p_h < p_t$ $(t > 1)$ be three points of S and let $q' < q$ be two points on the bend-line of p_t. Segments $\overline{p_h q}$ and $\overline{p_{h'} q'}$ do not intersect each other.

3.2 Computing 1-bend point-set embeddings

We describe a drawing algorithm, called `1-bend Universal Drawer`, that receives as input a planar graph G with n vertices and a necklace S of n points and returns a point-set embedding of G on S such that every edge of G is drawn with at most one bend. Algorithm `1-bend Universal Drawer` consists of the following steps.

Step 1: Compute a proper monotone topological book embedding Γ of G and the corresponding augmented proper topological book embedding Γ'. Let ℓ be the spine of Γ'. Label the real vertices of Γ' on ℓ by $v_{n-1}, \ldots, v_0$ in that order from left to right (*i.e.*, $v_i < v_{i-1}$). Map each real vertex v_i to point p_i of the necklace $(0 \leqslant i \leqslant n-1)$.

Step 2: Draw the bends of the arcs of the top page of Γ' as follows. For each vertex v_i of Γ' mapped to point p_i $(0 \leqslant i \leqslant n-1)$ do the following. Let $a_{i0}, a_{i1}, \cdots, a_{i(k-1)}$ be the sequence of arcs in the top page of Γ' whose right endpoint is v_i; assume that $a_{i0}, a_{i1}, \cdots, a_{i(k-1)}$ are encountered in this order when going clockwise around v_i by starting the tour from a point on ℓ slightly to the left of v_i. For each a_{ij} $(0 \leqslant j \leqslant k-1)$ do:

- Draw a ray r_{ij} emanating from p_i such that: (i) r_{ij} is inside the cone $C(p_i)$ of p_i, and (ii) $r_{i(j+1)}$ is to the right of r_{ij} $(0 \leqslant j \leqslant k-2)$.
- Let v_h be the left endpoint of a_{ij} in Γ' and b_{ij} the bend-point of a_{ij}. If v_h is a real vertex of Γ', draw b_{ij} at the intersection point, q, between r_{ij} and the bend-line of p_h (through p_{h-1}).[4] Else, if v_h is a division vertex of Γ' and the two real vertices bounding v_h in Γ' are v_t and v_{t-1}, draw b_{ij} at the intersection point, q, between r_{ij} and the bend-line of p_t.

Step 3: Draw the division vertices of Γ' as follows. For each division vertex d of Γ', do the following. Let (v_i, d) and (d, v_j) be the two arcs of Γ' sharing d such that (v_i, d) is in the bottom page and (d, v_j) is in the top page. Let q be the point computed in Step 2 such that q represents the bend of (d, v_j). Draw d at the intersection point between $\overline{p_i q}$ and $CH(S)$.

Step 4: Draw the arcs of Γ' as follows. For each arc (u, v) of Γ' do the following. Let p_u, p_v be the points representing u and v along $CH(S)$.

- If (u, v) is an arc in the bottom page, draw it as the chord $\overline{p_u p_v}$.

[4] If p_h and p_i are consecutive vertices of S $(h-1 = i)$, the ray r_{ij} and the bend-line of p_h do not intersect, though their closures intersect at p_i. For consistency, we draw b_{ij} at this intersection point $q = p_i$. In Step 4, the arc (v_h, v_i) is drawn as the poly-line consisting of segment $\overline{p_h q}$ followed by $\overline{q p_i}$, which is reduced to point p_i.

– If (u, v) is an arc in the top page of Γ', let q be the point computed at Step 2 that represents the bend-point of (u, v). Draw (u, v) as the poly-line consisting of segment $\overline{p_u q}$ followed by $\overline{q p_v}$.

Step 5: Let $\hat{\Gamma}$ be the drawing computed at the end of Step 4. Compute a drawing of G by removing from $\hat{\Gamma}$ those points that represent the division vertices of Γ'.

The proof of Theorem 1 is now completed by showing that Algorithm `1-bend Universal Drawer` correctly computes a point-set embedding of G on S such that each edge has at most one bend. The idea is to show that the drawing computed at the end of Step 5 maintains the topology of Γ and that the geometric properties of the proper monotone topological book embedding and of the necklace make it possible to point-set embed the graph without edge-crossings and with at most one bend per edge. In particular, we show that $\hat{\Gamma}$ is a planar drawing by exploiting Properties 1-3; the proof is however omitted here due to lack of space.

Observe that every real vertex of Γ' is drawn as a point of S in $\hat{\Gamma}$. Since $\hat{\Gamma}$ does not have edge crossings, removing the division vertices from $\hat{\Gamma}$ gives a point-set embedding of G on S. Also, by construction, the two edges incident on a division vertex of $\hat{\Gamma}$ form a flat angle, and thus removing the division vertices from $\hat{\Gamma}$ does not increase the number of bends. It follows that the drawing computed by Algorithm `1-bend Universal Drawer` is a point-set embedding of G on S such that each edge has at most one bend. Therefore, any necklace of n vertices is a 1-bend universal set for all planar graphs having n vertices, which concludes the proof of Theorem 1. We omit here the proofs on the size of the drawings.

4 Conclusion

We leave as an open problem to find a universal point-set for one-bend drawing of planar graphs in a polynomial-size regular grid.

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